INFLUENCE OF AIR FLOW ON THE ACOUSTIC CHARACTERISTICS
OF AN EXPANSION CHAMBER FILTER

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INFLUENCE OF AIR FLOW ON THE ACOUSTIC CHARACTERISTICS OF AN EXPANSION CHAMBER FILTER

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ABSTRACT. A theoretical and experimental study of the influence of air flow on the attenuation characteristics of a single expansion chamber filter is presented. The acoustic energy dissipation caused by air flow at the open end is counted for the equivalent resistance $\rho_0 cM$ which derived from Equations (1) - (4), where $\rho_0 c$ is the characteristic impedance of air and M is the Mach number of air flow. From Equations (6) - (8) and Equations (14) - (16), one gets the equations for the particle velocity at each area change in a tube with air flow, which was described by Equation (9) and Equation (17), and the effective cross section ratios σ_M and $\sigma_{M'}$ are obtained. Assuming that the area changes of a single expansion chamber filter can be expressed by these effective cross-section ratios, the attenuation characteristics of the filter with air flow for a reflection-free source and a constant-pressure source can be given by Equation (28) and Equation (31), respectively.

The theoretical values of the sound attenuation for a constant pressure source are compared with the measured values over a range of mean-flow velocity $U \leq 40$ m/sec. The results are shown in Figure 7.

1. Introduction

Research concerning acoustic filters was performed by persons such as Stewart [1], Mason [2], Lindsay [3], and Kobayashi [4] during the early period of electroacoustics. An analysis of them relied entirely on the similarities with distributed constant circuits or four-terminal circuits, which were already known in electric circuitry. In 1953, the results of systematic research were published concerning a series of silencers intended for engines by Don D. Davis et al. [5], and acoustic filters came to be widely known as silencers. However, the theoretical manner in which they were handled followed the same method adopted for acoustic filters in the past. This was the theory of so-called static characteristics, in which the effects of air flow were ignored. Ever since then, silencers have been designed by a method fundamentally based on the conventional mode of thinking [6, 7]. However, the effects of air flow cannot be ignored in engine mufflers, which are often used as silencers, and these mufflers also are structurally complex. For this reason, at the present time they are being designed using entirely empirical techniques.

Research concerning the effects of air flow on the propagation of sound waves or concerning the interaction between sound waves and air flow has in recent years been very extensive [8 - 11]. However, research concerning silencers has not yet been sufficient, and the results of future research have been awaited.

This paper is an analysis of the effects of air flow on the acoustical characteristics of an expansion chamber filter.

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*Numbers in the margin indicate pagination in the original foreign text.

The purpose of the research was to clarify the influence of air flow on silencers, which is regarded as of importance also from the practical standpoint. The fundamental ideas are the same as those adopted for resonance filters, which were dealt with in a previously published paper [12].

2. Acoustic Impedance at the Open End
where Air Flow is Ejected

Sound waves which have been propagated inside a tube with a sufficiently small diameter in comparison to the wavelength may be considered to be totally reflected at the open end. However, in cases when an air flow is being ejected from the open end, the reflection will not be total, and the sound waves will be radiated outside the tube at a rate in direct proportion to the air flow velocity [13].

Let us suppose that the particle velocity near the open end inside a tube with a sufficiently small diameter in comparison to the wavelength is $\dot{\xi}$, that the sectional area of the tube is S, and that the mean density of the air is ρ_0, as shown in Figure 1. In this case, the air near the open end can be represented by a piston having a mass $\rho_0 S \delta l$ and a vibration speed $\dot{\xi}$.

Here, δl is the piston length; it is sufficiently small in comparison with the wavelength.

If it is assumed that this piston is ejected outside of the tube by the air flow at a speed U (m/sec), as shown in Figure 1, the micromass put out within the
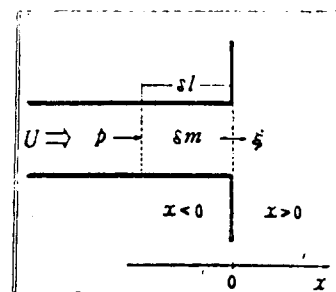

Figure 1. Diagram of a tube with an open end

3.

region where x > 0 within the micro-time interval δt will be:

$$\delta m = \rho_0 S U \delta t$$ (1)

Therefore, if the piston energy ejected out of the open end is E (Joule), this can be expressed as follows:

$$\frac{\partial E}{\partial t} = \frac{1}{2} \cdot \rho_0 S U \dot{\xi}^2 \quad \text{(Joule/sec)}$$ (2)

Thus, if Equation (2) is considered to be the dissipation function, the kinetic equation concerning this piston will be as follows:

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{\xi}}\right) + \frac{\partial}{\partial \dot{\xi}}\left(\frac{\partial E}{\partial t}\right) = S p \quad (N)$$ (3)

The T in Equation (3) represents the kinetic energy, while p represents the driving sound pressure towards the piston.

Consequently, based on Equation (2) and Equation (3), the impedance density of the piston viewed from inside the tube will be:

$$Z = \frac{p}{\dot{\xi}} = \rho_0 c M - i\omega\rho_0\delta l \quad \text{(MKS Rayls)}$$ (4)

Here, c represents the propagation speed of the sound waves, M — the Mach number of the air flow (U/c), and ω — the angular frequency of the driving sound pressure. It is assumed that $i = \sqrt{-1}$. When M = 0, Equation (4) is nothing but the impedance density when the open part is viewed from a position δl from the open end. Therefore, the acoustic impedance density at the open end where the air flow is ejected will be expressed as follows:

$$\rho_0 c M \quad \text{(MKS Rayls)}$$ (5)

3. Effects of Air Flow on Expanded Tube

When sound waves are propagated inside an acoustic tube in
which the sectional area of the tube varies discontinuously,
reflection of the sound waves occurs at discontinuous parts of
the tube. Let us next study how the propagation of these sound
waves is affected by the air flow moving through the inside of
the tube.

Figure 2 shows a tube A with a sectional area of S_1 which is
connected at x = 0 to a tube B having a sectional area of S_2.
If the incident sound pressure, the re-
flected sound pressure, and the trans-
mitted sound pressure are represented as
p_1, p_2, and p_3, and if the particle
velocities corresponding to them are re-
presented as $\dot{\xi}_1$, $\dot{\xi}_2$, and $\dot{\xi}_3$, the fol-
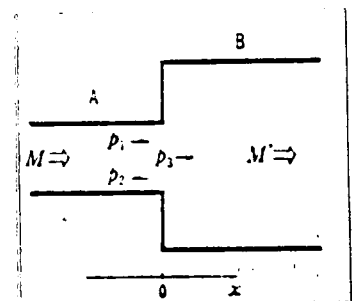lowing will hold at x = 0:

$$p_1 + p_2 = p_3 \quad (N/m^2)$$ (6)

Figure 2. Diagram
of a tube A and B
connecting at x = 0

On the other hand, if the air inside
tube A in the vicinity of the connection
surface of both tubes is approximated by a piston, as described
in the preceding section, since the equivalent resistance density
caused by the air flow is expressed by Equation (5), the rela-
tionship between the sound pressure and the particle speed at
x = 0 will be:

$$\frac{p_1 + p_2}{\dot{\xi}_1 - \dot{\xi}_2} = \left(\frac{S_1 p_3}{S_2 \dot{\xi}_3}\right)_{M=0} + \rho_0 c M \quad \text{(MKS Rayls)}$$ (7)

The first term on the right side of Equation (7) is the acoustic
impedance density in cases when there is no flow, and the second

term is the equivalent resistance density caused by the air
flow. Concerning these sound pressures and particle velocities,
if we make use of the following relationships:

$$\left.\begin{array}{l} p_1/\xi_1 = \rho_0 c(1+M), \\ p_2/\xi_2 = \rho_0 c(1-M), \\ (p_3/\xi_3)_{M=0} = \rho_0 c, \quad p_3/\xi_3 = \rho_0 c(1+M') \end{array}\right\} \quad (8)$$

the equation for the particle velocity at x = 0 will be as
follows, from Equation (6) and Equation (7):

$$\begin{array}{l} \xi_1 - \xi_2 = \dfrac{\sigma}{1+\sigma M}(\xi_3)_{M=0} \\ \qquad = \left(\dfrac{1+M'}{1+\sigma M}\right)\sigma \xi_3 \quad \text{(m/sec)} \end{array} \quad (9)$$

Here, σ is the sectional area ratio of both tubes ($S_2/S_1 < 1$),
and M' represents the Mach number of the air flow inside tube B.
Here, if we represent the effective sectional area ratio when
there is air flow as follows:

$$\sigma_M = \frac{1+M'}{1+\sigma M}\cdot \sigma \quad (10)$$

Equation (9) will become formally the same as the well known
equation for the particle velocity when there is no air flow.

From Equations (6) - (8), the sound pressure reflection
coefficient will be:

$$r_M = \frac{p_2}{p_1} = \left(\frac{1-M}{1+M}\right)\left(\frac{1-\sigma}{1+\sigma}\right) \quad (11)$$

4. Effects of Air Flow on Contracted Tube

If the sound field inside the tube can be expressed approxi-
mately by the (0, 0) mode alone, the air inside tube B near the
connecting surface of both tubes can be approximated by a piston
in the same way as in the preceding section, as shown in Figure 3.

If the sectional area of tube B is S_2, the sectional area of tube C is S_3, the sectional area ratio of both tubes is $\sigma = S_2/S_3$, and the Mach number of the air flow inside tube C is M, the Mach number of the air flow inside tube B will be $M' = M/\sigma$. However, the air flow near the open surface of tube B in the vicinity of $x = l$ will have a flow velocity closer to the flow velocity inside tube C. Consequently, if the Mach number of the air flow in this section is expressed as M, the equivalent resistance of the hypothetical piston can be expressed as follows:
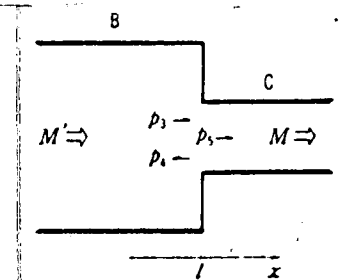
Figure 3. Diagram of a tube B and C connecting at $x = l$

$$\rho_0 c M S_2 \tag{12}$$

Therefore, the equivalent resistance density produced by the air flow in the connection part, viewed from tube B, will be:

$$\rho_0 c M S_3/S_2 = \rho_0 c M'' \quad \text{(MKS Rayls)} \tag{13}$$

Consequently, if the sound pressures of the incident waves, the reflected waves, and the transmitted waves are p_3, p_4, and p_5, and the particle velocities corresponding to them are $\dot{\xi}_3$, $\dot{\xi}_4$, and $\dot{\xi}_5$, the following will be obtained at $x = l$:

$$p_3 + p_4 = p_5 \quad (N/m^2), \tag{14}$$

$$\frac{p_3 + p_4}{\dot{\xi}_3 - \dot{\xi}_4} = \left(\frac{S_2 p_5}{S_3 \dot{\xi}_5}\right)_{M=0} + \rho_0 c M' \quad \text{(MKS Rayls)} \tag{15}$$

On the other hand, since the following relationship holds between the sound pressure and the particle velocity:
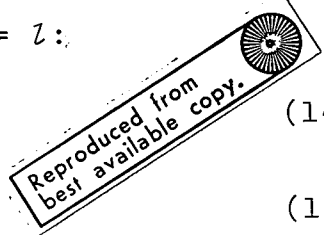
$$p_3/\xi_3 = \rho_0 c(1 + M'),$$
$$p_4/\xi_4 = \rho_0 c(1 - M'),$$
$$(p_5/\xi_5)_{M=0} = \rho_0 c, \quad p_5/\xi_5 = \rho_0 c(1 + M)$$ (16)

in view of Equations (14) - (16), the equation with reference
to the particle velocity will be expressed as follows at x = l:

$$\dot\xi_3 - \dot\xi_4 = (\dot\xi_5)_{M=0}/\sigma(1 + M'/\sigma)$$
$$= \dot\xi_5/\sigma_{M'} \quad (m/sec)$$ (17)

At the effective sectional area ratio of the contracted tube
when there is air flow, $\sigma_{M'}$ will be as follows:

$$\sigma_{M'} = \frac{1 + M'/\sigma}{1 + M} \cdot \sigma$$ (18)

Also, in view of Equations (14) - (16), the sound pressure re-
flection coefficient will be as follows:

$$r_{M'} = \frac{p_4}{p_3} = \left(\frac{1 - M'}{1 + M'}\right)\left(\frac{\sigma - 1}{\sigma + 1}\right)$$ (19)

5. Effects of Air Flow on Expansion Chamber Filter

The sectional area ratio of an expanded tube from which air
flow is ejected, shown in Equation (10), will become equi-
valently smaller on account of the air flow and, if M' << 1, the
effective sectional area ratio will be:

$$\sigma_M \approx \frac{\sigma}{1 + \sigma M}$$ (20)

On the other hand, the effects of air flow will be smaller in
a contracted tube than in an expanded tube, and if M' << 1,
the effective sectional area ratio will be as follows due to
Equation (18):

$$\sigma_{M'} \approx \frac{\sigma}{1 + M}$$ (21)

If it is assumed that the discontinuous sectional area
ratios of an expansion chamber filter can be expressed by this
type of effective sectional area ratio, the equation concerning

the sound pressure and the particle velocity, when there is flow, for an acoustic tube such as that shown in Figure 4 can be represented as follows when x = 0:

$$p_1 + p_2 = p_3 + p_4 \quad (\text{N/m}^2), \tag{22}$$

$$\xi_1 - \xi_2 = \sigma_M(\xi_3 - \xi_4) \quad (\text{m/sec}). \tag{23}$$
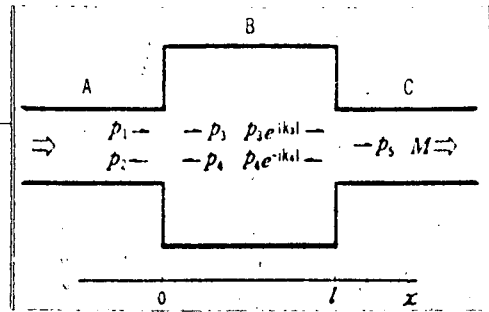
It can be expressed as follows when x = l:

$$p_3 \exp(ik_3 l) + p_4 \exp(-ik_4 l) = p_5 \quad (\text{N/m}^2), \tag{24}$$

$$\xi_3 \exp(ik_3 l) - \xi_4 \exp(-ik_4 l) = \xi_5/\sigma_M \quad (\text{m/sec}). \tag{25}$$

Figure 4. Diagram of a single expansion chamber filter

Here, k_3 and k_4 are the wavelength constants of the sound waves propagated in the positive and negative directions of x inside tube B. If k = ω/c, they can be expressed as follows:

$$k_3 = k/(1+M'), \quad k_4 = k/(1-M'). \tag{26}$$

Using the relationships in Equation (8) and Equation (16), let us seek the ratio between the incident sound pressure and the transmitted sound pressure by means of Equations (22) - (25):

$$\frac{p_1}{p_5} = \frac{1-M'^2}{4\sigma_{M'}}\left\{\left(1+\frac{1-M}{1+M'}\sigma_M\right)\left(1+\frac{1+M}{1-M'}\sigma_{M'}\right)\right.$$
$$\cdot \exp(-ik_3 l) - \left(1-\frac{1-M}{1-M'}\sigma_M\right)$$
$$\left. \cdot\left(1-\frac{1+M}{1+M'}\sigma_{M'}\right)\exp(ik_4 l)\right\}. \tag{27}$$

Consequently, the absolute value in Equation (22) can be expressed approximately as follows:

$$\left|\frac{p_1}{p_5}\right| \simeq \frac{1}{2}\left\{\left[(1-M)\frac{\sigma_M}{\sigma_{M'}}+(1+M)\right]^2\right.$$
$$+\frac{\sigma_M}{\sigma_{M'}}\left[(1-M)\sigma_M-\frac{1}{(1-M)\sigma_M}\right]$$
$$\left.\cdot\left[(1+M)\sigma_{M'}-\frac{1}{(1+M)\sigma_{M'}}\right]\sin^2(kl)\right\}^{1/2}$$
$$(M', M^2 \ll 1) \tag{28}$$

The sound pressure attenuation can be expressed as $20 \log_{10}$ $|p_1/p_5|$. If it is assumed that $\sigma_{M2} \gg 1$, the attenuation will be approximately as follows:

$$20 \log_{10}\left|\frac{p_1}{p_5}\right| \simeq 10 \log_{10}\left[1+\left(\frac{\sigma_M}{2}\right)^2 \sin^2(kl)\right]$$
$$(\text{dB}) \tag{29}$$

Equations (28) and (29) indicate that the sound pressure attenuation of an expansion chamber filter will grow less as the flow velocity increases.

6. Experiments

6.1. Experimental Equipment and Measuring Method

The experimental equipment is shown in Figure 5. The side of the sound source includes a speaker for generating pure tones, and a blower is installed after a muffler. It is possible to obtain with this blower a flow velocity of up to about 50 m/sec in the duct section. On the other hand, on the sound receiving side there is a horn with a cut-off frequency of 70 Hz. In the measuring frequency band zone, the termination is almost reflection-free without a connection with the velocity of the air flow. The duct at the section where the muffler is installed is made of wood with a thickness of 50 mm. It is built so that vibrations from the sound source will not be transmitted to the sound receiving side.
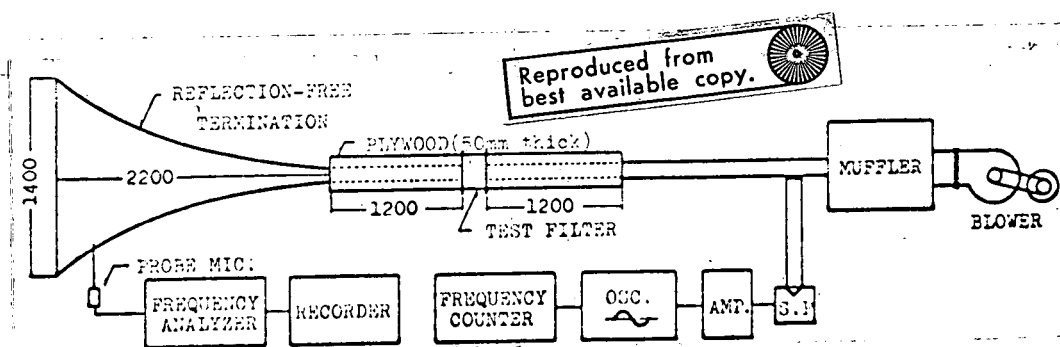
Figure 5. Experimental arrangement for measuring
attenuation due to a single expansion chamber
filter with air flow

The expansion chamber filter, as shown in Figure 6, is
made of a steel plate with a sectional area ratio of $\sigma = 9$ and
a cavity length of $l = 500$ mm. The steel plates in the cavity
section are provided with vibration damping materials, so that
plate vibrations will not occur. During the experiments, fur-
ther reinforcements were provided from the outside to prevent
vibrations.

The sound pressure on the sound receiving side when the
muffler is attached and when it is removed is measured by the
probe microphone inserted on the side wall of the horn. It is
passed through a narrow
band frequency analyzer
and recorded. The
noises produced by the
air flow are attenuated
by means of the narrow
band frequency analyzer
at a rate of 45 dB/
octave with reference to
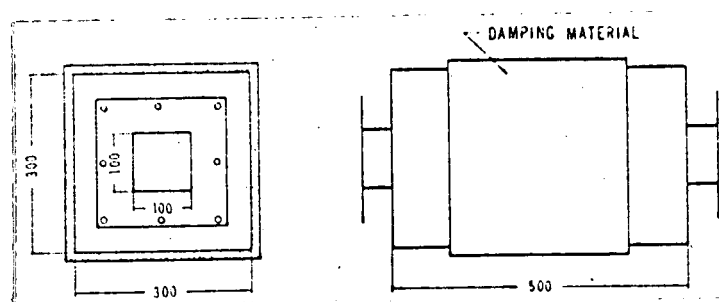the pass band frequency;
they are separated from

Figure 6. Schematic diagram of
a single expansion chamber for
attenuation tests

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the pure tone emitted from the sound source. However, since the noises produced by the air flow become larger as the flow velocity increases, it is difficult to separate them. In this experimental equipment, when the average flow velocity was 40 m/sec, it was impossible to make adequate measurements of the sound pressure at frequencies below 300 Hz.

The measured value of the sound pressure attenuation is the difference in the sound pressure level on the sound receiving side due to the presence or absence of a muffler. However, its value differs depending upon the conditions at the sound source, and the attenuation expressed by Equation (29) is that produced when the impedance on the sound source side is $\rho_0 c$ — that is,

when the sound source side, as well as the sound receiving side, is a reflection-free end. On the other hand, when the sound source side is not a reflection-free end — that is, when the impedance on the sound source side as a limit is either infinitely great or zero — is is necessary to treat the sound source either as a constant-speed sound source or as a source of constant sound pressure, respectively. An automobile engine is regarded as one of the representative examples of a constant-speed sound source accompanied by air flow. However, since it is difficult to separate the sounds coming from the sound source from the sounds produced by the air flow, it cannot be used for the purpose of making direct comparisons with the theoretical value of the attenuation. The sound source side of the experimental equipment used in these experiments was remodeled so that the sound pressure attenuation at conditions of no air flow in the preliminary experiments would be as close as possible to the theoretical characteristics in the case of a source of constant sound pressure.

6.2. Comparison of Theoretical and Measured Values

The sound pressure attenuation of an expansion chamber filter with respect to a source of constant sound pressure, when the sound receiving side is a reflection-free termination, is expressed in the following manner, if the sound pressure on the receiving side when there is no muffler is P_0, and the sound pressure on the sound receiving side when a muffler has been installed is p_5:

$$20 \log_{10} \left| \frac{p_0}{p_5} \right| \quad (dB) \qquad (30)$$

However, from Equations (22) - (25), one obtains the following:

$$\left| \frac{p_0}{p_5} \right| = \left| \frac{p_1 \exp(-ik_1 L) + p_2 \exp(ik_2 L)}{p_5} \right|$$
$$\simeq \{ \sigma_M{}^2 \sin^2(kl) \sin^2(kL') - 2\sigma_M \sin(kl)$$
$$\cdot \cos(kl) \sin(kL') \cos(kL') + \cos^2(kl) \}^{1/2}$$
$$(L' = L/(1-M^2) \simeq L, \ M'^2, \ M^2 \ll 1). \ (31)^* \qquad (31)^{[1]}$$

Here L represents the distance from the expansion chamber filter to the constant sound pressure sound source. The apparent position of the constant sound pressure sound source in this experimental equipment is about 1.5 m from the muffler, the place where the acoustic tube coming from the speaker enters the duct. It corresponds to the place where the sectional area of the duct is expanded, viewed from the muffler side.

Figure 7 gives the theoretical values and measured values of the sound pressure attenuation characteristics of the expansion chamber filter with respect to a constant sound pressure sound source. The changes in the attenuation characteristics from a state with no air flow up to a flow velocity of 40 m/sec

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[1] Refer to appendix.

are shown. Since the measured
value of the attenuation at a
state of no air flow is
rather close to the theoreti-
cal characteristics with
respect to a constant sound
pressure sound source, it
is possible to regard the
sound source of this experi-
mental equipment more or
less as a constant sound
pressure sound source.
The sound pressure attenua-
tion declines gradually as
the flow velocity increases.
For instance, the theoreti-
cal value of the maximum
attenuation in the vicinity
of 170 Hz at a state of no
air flow is 19.1 dB. How-
ever, when the velocity of
the air flow is 10 m/sec,
it is 17.0 dB; when the
velocity is 20 m/sec, it
decreases to 15.4 dB. When
the velocity is 30 m/sec,
it decreases to 14.1 dB,

Figure 7. Attenuation character-
istics of a single expansion
chamber filter (―― calculated
from Equation 31, o measured):
1 ― without air flow; 2 ― mean-
flow velw velocity U = 10 m/sec,
3U = 20 m/sec, 4U = 30 m/sec,
5U = 40 m/sec

and when the velocity is 40 m/sec, it decreases to 12.8 dB.
These changes in the sound pressure attenuation depending on
increases in the flow velocity are clearly apparent in the meas-
urement results, if one excludes certain local variations. The
maximum value of the attenuation in the vicinity of 170 Hz at
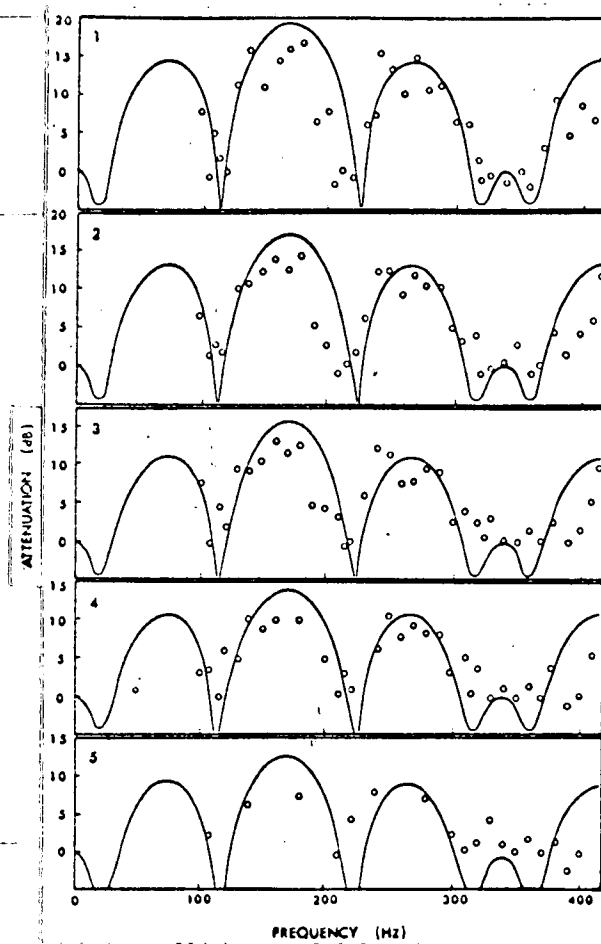a state of no air flow is 17 dB; at a flow velocity of 10 m/sec,

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it is 14.5 dB; at a flow velocity of 20 m/sec, it is 13 dB; and at a flow velocity of 30 m/sec, it is around 10 dB.

The differences between the measured values and the theoretical values are due in part to the differences between the experimental equipment and the conditions assumed in the theoretical equations. They are also partly attributable to differences between physical phenomena and the simplified theory. As for the former item, problems remain chiefly in relation to improvements of the sound source side. As for the latter, problems arise when the air flow becomes more rapid.

7. Conclusion

Since it is difficult to treat mathematically the air flow itself which plows along inside the muffler, it becomes necessary to simplify the phenomena considerably in order to express the propagation of sound waves inside the muffler accompanied by air flow. Naturally, the validity of the results is to be judged according to the experimental results. However, in many cases, as the air flow becomes more rapid, it becomes difficult even to carry out such experiments themselves. Here, we merely gave the experimental results up to a flow velocity of 40 m/sec. However, one can easily assume, even from the results of this research, that the characteristics of the mufflers installed on the engines will differ from static characteristics.

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APPENDIX

Approximate calculations for Equation (31):

From Equations (22) - (25), we obtain

$$\frac{p_2}{p_s} = \left(\frac{1-M}{1+M}\right) \cdot \frac{1-M'^2}{4\sigma_{M'}} \left\{\left(1 - \frac{1+M}{1+M'}\sigma_M\right)\right.$$
$$\cdot \left(1 + \frac{1+M}{1-M'}\sigma_{M'}\right) \exp(-ik_2l)$$
$$- \left(1 + \frac{1+M}{1-M'}\sigma_M\right)\left(1 - \frac{1+M}{1+M'}\sigma_{M'}\right)$$
$$\left. \cdot \exp(ik_1l)\right\} \qquad\qquad\qquad (A.1)$$

Thus, from Equation (27) and Equation (A.1), we obtain

$$\frac{p_1 \exp(-ik_1L) + p_2 \exp(ik_2L)}{p_s} = \frac{1-M'^2}{4\sigma_{M'}}$$
$$\{A \exp(-ikl) - B \exp(ikl)\} \exp(ikM'l$$
$$-ik_2L) + \left(\frac{1-M}{1+M}\right) \cdot \frac{1-M'^2}{4\sigma_{M'}} \cdot \{C \exp(-ikl)$$
$$- D \exp(ikl)\} \exp(ikM'l + ik_2L) \qquad (A.2)$$

Here,

$$A = \left(1 + \frac{1-M}{1+M'}\sigma_M\right)\left(1 + \frac{1+M}{1-M'}\sigma_{M'}\right),$$
$$B = \left(1 - \frac{1-M}{1-M'}\sigma_M\right)\left(1 - \frac{1+M}{1+M'}\sigma_{M'}\right),$$
$$C = \left(1 - \frac{1+M}{1+M'}\sigma_M\right)\left(1 + \frac{1+M}{1-M'}\sigma_{M'}\right),$$
$$D = \left(1 + \frac{1+M}{1-M'}\sigma_M\right)\left(1 - \frac{1+M}{1+M'}\sigma_{M'}\right).$$

Consequently, if M'^2, $M^2 \ll 1$, the following expression is valid:

$$\left|\frac{p_1 \exp(-ik_1L) + p_2 \exp(ik_2L)}{p_s}\right|$$
$$\simeq \{\sigma_M^2 \sin^2(kl) \sin^2(kL') - 2\sigma_M \sin(kl)$$
$$\cdot \cos(kl) \sin(kL') \cos(kL') + \cos^2(kl)\}^{1/2} \qquad (A.3)$$

Here, $L' = L/(1 - M^2) \simeq L$.